About the Rationality of the Knizhnik-Zamolodchikov Equation Solution

Lev Sakhnovich

735 Crawford ave., Brooklyn, 11223, New York, USA.
E-mail address: lev.sakhnovich@verizon.net

Abstract

In the paper the solution of KZ system (n=4, m=2) is constructed in the explicit form in terms of the hypergeometric functions. We proved that the corresponding solution is rational when the parameter ρ is integer. We show that in the case (n=4, m=5, ρ is integer) the corresponding KZ system hasn't got a rational solution.

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1 Introduction

1. We consider the differential system

$$\frac{\partial W(z_1, z_2, ..., z_n)}{\partial z_k} = \rho A_k(z_1, z_2, ...z_n)W, \quad 1 \leq j \leq n, \tag{1.1}$$

where $A_k(z_1, z_2, ..., z_n)$ and $W(z_1, z_2, ..., z_n)$ are $m \times m$ matrix functions. We suppose that $A_k(z_1, z_2, ..., z_n)$ has the form

$$A_k(z_1, z_2, ..., z_n) = \sum_{j=1, j\neq k}^{n} \frac{P_{k,j}}{z_k - z_j}, \tag{1.2}$$

where $z_k \neq z_j$ if $k \neq j$. Here the matrices $P_{j,k}$ are connected with the matrix representation of the symmetric group S_n and are defined by formulas (2.1)-(2.4). We note that the well-known Knizhnik- Zamolodchikov equation has the form (1.1), (1.2) (see [4]). This system has found applications in several areas of mathematics and physics (see [4],[12]). In paper ([11],section 2) we prove the following assertion:

Theorem 1.1. *The fundamental solution of KZ system (1.1),(1.2) is rational,when ρ is integer and matrices $P_{k,j}$ give the natural representation of symmetric group S_n.*

We note that the natural representation is the sum of the 1-representation and the irreducible (n-1)-representation. We name the corresponding irreducible (n-1)-representation *the natural representation* as well.

In the present paper the solution of KZ system (n=4, m=2) is constructed in the explicit form in the terms of the hypergeometric functions. We proved that the corresponding solution is rational,if ρ is integer. A number of authors (see [3],[5]) affirmed that the solution of KZ equation is rational for all the representation of symmetric group S_n, if the parameter ρ is integer. In this paper we construct the example (n=4 and n=5) for which ρ is integer but the corresponding solution is irrational. This result leads to the following open problem.

Open Problem 1.1 *Let ρ be integer and let $P_{k,j}$ be an irreducible representation of symmetric group S_n. To find the conditions under which the corresponding fundamental solution is rational.*

Here can be useful the following necessary condition see [9]):

Proposition 1.1 *If ρ is integer and the fundamental solution of system (1.1), (1.2) is rational then all the eigenvalues of matrices*

$$Q_k = \sum_{j \neq k, j=1}^{n} P_{k,j}, \quad 1 \leq k \leq n, \tag{1.3}$$

are integer.

In the before mentioned case (n=4, m=2) the formulated necessary condition is fulfilled.

2 Knizhnik-Zamolodchikov Equation Solution, Case n=4, m=2.

In the case n=4, m=2 we have the following matrix irreducible representation (see [2], [6]):

$$P_{1,2}=P_{3,4}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_{2,4}=P_{1,3}=\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \tag{2.1}$$

$$P_{2,3}=P_{1,4}=\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}. \tag{2.2}$$

Following A.Varchenko [12] we change the variables

$$u_1=z_1-z_2, \quad u_k=\frac{z_k-z_{k+1}}{z_{k-1}-z_k}, \quad (2\leq k\leq 3), \tag{2.3}$$

$$u_4=z_1+z_2+z_3+z_4. \tag{2.4}$$

The KZ system takes the following form

$$\frac{\partial W}{\partial u_j}=\rho H_j(u)W, \quad (1\leq j\leq 4), \tag{2.5}$$

where $u=(u_1,u_2,u_3,u_4)$. We have [11]:

$$\frac{\partial W}{\partial u_1}=\rho\left(\frac{\Omega_1}{u_1}\right)W, \tag{2.6}$$

$$\frac{\partial W}{\partial u_2}=\rho\left(\frac{\Omega_2}{u_2}+\frac{P_{1,3}}{1+u_2}+\frac{P_{1,4}(1+u_3)}{1+u_2+u_2u_3}\right)W, \tag{2.7}$$

$$\frac{\partial W}{\partial u_3}=\rho\left(\frac{P_{4,3}}{u_3}+\frac{P_{4,2}}{1+u_3}+\frac{P_{4,1}u_2}{1+u_2+u_2u_3}\right)W, \tag{2.8}$$

where $H_4(u)=0$ and

$$P_r=\sum_{j>r}P_{j,r}, \quad \Omega_s=P_s+P_{s+1}+...+P_4. \tag{2.9}$$

It follows from (2.1),(2.2) and (2.9) that $\Omega_1=\Omega_2=0$. Hence KZ system (2.7), (2.8) takes the form

$$\frac{\partial W}{\partial u_2}=\rho\left(\frac{P_{1,3}}{1+u_2}+\frac{P_{1,4}(1+u_3)}{1+u_2+u_2u_3}\right)W, \tag{2.10}$$

$$\frac{\partial W}{\partial u_3} = \rho\left(\frac{P_{4,3}}{u_3} + \frac{P_{4,2}}{1+u_3} + \frac{P_{4,1}u_2}{1+u_2+u_2u_3}\right)W. \tag{2.11}$$

We introduce the matrix function

$$F(y) = W(y)(1+y)^{\rho}(1+y+yz)^{\rho}, \quad y = u_2, \quad u_3 = z. \tag{2.12}$$

Relations (2.10) and (2.12) imply that

$$\frac{dF}{dy} = \rho\left(\frac{P_{1,3}+I}{1+y} + \frac{(P_{1,4}+I)(1+z)}{1+y+yz}\right)F. \tag{2.13}$$

We consider the constant vectors

$$w_1 = \mathrm{col}[\sqrt{3}, 1], \quad w_2 = \mathrm{col}[1, \sqrt{3}]. \tag{2.14}$$

It is easy to see that

$$(P_{1,3}+I)w_1 = 0, \quad (P_{1,3}+I)w_2 = -\sqrt{3}w_1 + 2w_2, \tag{2.15}$$

$$(P_{1,4}+I)w_1 = \sqrt{3}w_2, \quad (P_{1,4}+I)w_2 = 2w_2, \tag{2.16}$$

We represent $F(y)$ in the form

$$F(y) = \phi_1(y)w_1 + \phi_2(y)w_2 \tag{2.17}$$

and substitute it in (2.13). In view of (2.15) and (2.16) we have

$$\phi_1'(y) = -\frac{\sqrt{3}\rho}{1+y}\phi_2(y), \tag{2.18}$$

$$\phi_2'(y) = \rho\left(\frac{2\phi_2(y)}{1+y} + \frac{(2\phi_2(y)+\sqrt{3}\phi_1)(1+z)}{1+y+yz}\right). \tag{2.19}$$

$$(1+y)(\frac{1}{z+1}+y)\phi_1''(y) + B(z,y)\phi_1'(y) + 3\rho^2\phi_1(y) = 0, \tag{2.20}$$

where

$$B(z,y) = \frac{1}{z+1} + y - 2\rho[2(\frac{1}{z+1}+y) + \frac{z}{z+1}] \tag{2.21}$$

Changing the variable $y = \frac{-z}{z+1}v-1, \quad \phi(v) = \phi_1(\frac{-z}{z+1}v-1)$ we reduce equation (2.20) to the following form

$$v(1+v)\phi''(v) + [1+v-2\rho(1+2v)]\phi'(v) + 3\rho^2\phi(v) = 0. \tag{2.22}$$

By introducing $\psi(v) = \phi(-v)$ we obtain Gauss hypergeometric equation [1]:

$$v(1-v)\psi''(v) + [\gamma - (\alpha+\beta+1)y]\psi'(v) - \alpha\beta\psi(v) = 0, \tag{2.23}$$

where

$$\alpha = -\rho, \quad \beta = -3\rho, \quad \gamma = 1-2\rho. \tag{2.24}$$

In our paper [11] we have proved the following assertion.
Proposition 2.1. *The solutions of Gauss hypergeometric equation (2.23) are rational functions if*

$$\alpha = -\rho, \quad \beta = -3\rho, \quad \gamma = 1-2\rho,$$

where ρ is integer.
Let $\psi_1(v)$ and $\psi_2(v)$ be linearly independent rational solutions of equation (2.22). Then the vector functions

$$Y_k(y,z) = (1+y)^{-\rho}(1+y+yz)^{-\rho}X_k(y,z), \quad (k=1,2), \tag{2.25}$$

where

$$X_k(y,z) = \psi_k\left(\frac{(y+1)(z+1)}{z}\right)w_1 + \frac{(y+1)(z+1)}{z\sqrt{3}\rho}\psi_k'\left(\frac{(y+1)(z+1)}{z}\right)w_2 \tag{2.26}$$

are the solutions of system (2.13). Hence we deduced the assertion
Proposition 2.2. *The fundamental solution $W_1(y,z,\rho)$ of system (2.10) is defined by relation*

$$W_1(y,z,\rho) = [Y_1(y,z,\rho), Y_2(y,z,\rho)], \tag{2.27}$$

where $y = u_2, \quad z = u_3$.
Proposition 2.3. *If $\rho = -1$ then the solutions $Y_1(y,z,-1)$ and $Y_2(y,z,-1)$ of equation (2.10) have the forms*

$$Y_1(y,z,-1) = -z(y+1)w_1 + \frac{z(z+1)(y+1)^2}{\sqrt{3}(1+y+yz)}w_2, \tag{2.28}$$

$$\begin{aligned} Y_2(y,z,-1) = {} & [\frac{z^2(1+y+yz)}{(z+1)^2(y+1)} + \frac{z(1+y+yz)}{(z+1)}]w_1 \\ & -[\frac{2z^2(1+y+yz)}{\sqrt{3}(z+1)^2(y+1)} + \frac{z(1+y+yz)}{\sqrt{3}(z+1)}]w_2. \end{aligned} \tag{2.29}$$

3 Common solution, Consistency

We note that the KZ system (2.10), (2.11) is consistent [11]. In section 2 we have considered only the first equation of system (2.10), (2.11). Now using this result we shall construct the common solution of the KZ system (2.10),(2.11). To do it let us consider equation (2.11) in case when $u_2 = 0, \quad u_3 = z$. We have

$$\frac{\partial W_2}{\partial z} = \rho\Big(\frac{P_{4,3}}{z} + \frac{P_{4,2}}{1+z}\Big)W_2. \tag{3.1}$$

This equation can be solved in the same way as (2.10). We introduce the matrix function

$$G(z) = W_2(z)z^{\rho}(1+z)^{\rho}, \quad z = u_3. \tag{3.2}$$

Relations (3.1) and (3.2) imply that

$$\frac{dG}{dz} = \rho\Big(\frac{P_{4,3}+I}{z} + \frac{P_{4,2}+I}{1+z}\Big)G. \tag{3.3}$$

We consider the constant vectors

$$v_1 = \mathrm{col}[0,2], \quad v_2 = \mathrm{col}[1,-\sqrt{3}]. \tag{3.4}$$

It is easy to see that

$$(P_{4,3}+I)v_1 = 0, \quad (P_{4,2}+I)w_1 = -\sqrt{3}v_2, \tag{3.5}$$

$$(P_{4,3}+I)v_2 = \sqrt{3}v_1 + 2v_2, \quad (P_{4,2}+I)v_2 = 2v_2, \tag{3.6}$$

We represent $G(z)$ in the form

$$G(z) = \phi_1(z)v_1 + \phi_2(z)v_2 \tag{3.7}$$

and substitute it in (3.3). In view of (3.5) and (3.6) we have

$$\phi_1'(z) = \frac{\sqrt{3}\rho}{z}\phi_2(z), \quad \phi_2'(z) = \rho(\frac{2\phi_2(z)}{z} + \frac{2\phi_2(z)}{1+z} - \frac{\sqrt{3}\phi_1(z)}{1+z}). \tag{3.8}$$

It follows from (3.8) that

$$z(1+z)\phi_1''(z) + [1+z-2\rho(1+2z)]\phi_1'(z) + 3\rho^2\phi_1(z) = 0. \tag{3.9}$$

By introducing $\psi(z) = \phi_1(-z)$ we reduce equation (3.9) to Gauss hypergeometric equation (2.23), (2.24). Let $\psi_1(z)$ and $\psi_2(z)$ be linearly independent solutions of equation (2.23). Then the vector functions

$$U_k(z,\rho) = [\psi_k(-z)v_1 - \frac{z}{\sqrt{3}\rho}\psi_k'(-z)v_2]z^{-\rho}(1+z)^{-\rho}, \quad k = 1,2 \tag{3.10}$$

are the solutions of system (3.1). Hence we deduced the assertion
Proposition 3.1. *The fundamental solution $W_2(z,\rho)$ of system (3.1) is defined by the relation*

$$W_2(z,\rho) = [U_1(z,\rho), U_2(z,\rho)]. \tag{3.11}$$

Let us consider separately the case when $\rho = -1$. In this case equation (3.9) takes the form

$$z(1+z)\phi_1''(z) + (3+5z)\phi_1'(z) + 3\phi_1(z) = 0. \tag{3.12}$$

It is easy to check by direct calculation that the functions

$$\phi_{1,1} = \frac{1}{1+z}, \quad \phi_{1,2} = \frac{1-z}{z^2} \tag{3.13}$$

are solutions of equation (3.12). Hence the following assertion is true:
Proposition 3.2. *If $\rho = -1$ then the solutions $U_1(z,-1)$ and $U_2(z,-1)$ of equation (3.12) have the forms*

$$U_1(z,-1) = zv_1 + \frac{z^2}{\sqrt{3}(1+z)}v_2, \quad U_2(z,-1) = \frac{1-z^2}{z}v_1 - \frac{(z-2)(z+1)}{\sqrt{3}z}v_2 \tag{3.14}$$

We have constructed the fundamental solutions $W_1(y,z,\rho)$ and $W_2(y,z,\rho)$ of equations (2.10) and (3.1) respectively. Now using our previous results (see[],section 3) we obtain the main theorem of the present paper.
Theorem 3.1. *The 2×2 matrix function*

$$W(y,z,\rho) = W_1(y,z,\rho)W_1^{-1}(0,z,\rho)W_2(0,z,\rho) \tag{3.15}$$

is the fundamental solution of the KZ system (2.10),(2.11), where $y = u_2, \quad z = u_3$. If ρ is integer this fundamental solution is rational.

4 System with Non-rational Solution.

Let us consider the case n=5, m=5.
In this case we have the following matrix irreducible representation (see [2], [6]):

$$P_{1,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.1}$$

$$P_{1,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix}, \tag{4.2}$$

$$P(1,4) = \begin{pmatrix} -1/3 & -\sqrt{2}/3 & -\sqrt{6}/3 & 0 & 0 \\ -\sqrt{2}/3 & 5/6 & -\sqrt{3}/6 & 0 & 0 \\ -\sqrt{6}/3 & -\sqrt{3}/6 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & \sqrt{3}/2 \\ 0 & 0 & 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}, \tag{4.3}$$

$$P(1,5) = \begin{pmatrix} -1/3 & \sqrt{2}/9 & \sqrt{6}/9 & -4/9 & -4\sqrt{3}/9 \\ \sqrt{2}/9 & -19/54 & 23\sqrt{3}/54 & -8\sqrt{2}/27 & 4\sqrt{6}/27 \\ \sqrt{6}/9 & 23\sqrt{3}/54 & 1/2 & 4\sqrt{6}/27 & 0 \\ -4/9 & -8\sqrt{2}/27 & 4\sqrt{6}/27 & 37/54 & -5\sqrt{3}/54 \\ -4\sqrt{3}/9 & 4\sqrt{6}/27 & 0 & -5\sqrt{3}/54 & 1/2 \end{pmatrix}. \tag{4.4}$$

Using the previous representations of $P(1,k)$ we deduce that the matrix $Q_1 = P(1,2) + P(1,3) + P(1,4) + P(1,5)$ has the form

$$T_{-1} = \begin{pmatrix} C_1 & C_2 & \dots & C_5 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 4/3 \\ -2\sqrt{2}/9 \\ -(2\sqrt{2/3})/9 \\ -4/9 \\ -4/(3\sqrt{3}) \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -2\sqrt{2}/9 \\ 53/54 \\ 7/(9\sqrt{3}) - \sqrt{3}/2 \\ -8\sqrt{2}/27 \\ (4\sqrt{2/3})/9 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} -(2\sqrt{2/3})/3 \\ 7/(9\sqrt{3}) - \sqrt{3}/2 \\ 1/2 \\ (4\sqrt{2/3})/9 \\ 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -4/9 \\ -8\sqrt{2}/27 \\ (4\sqrt{2/3})/9 \\ 37/54 \\ -5/(18\sqrt{3}) \end{pmatrix}, \qquad C_5 = \begin{pmatrix} -4/(3\sqrt{3}) \\ (4\sqrt{2/3})/9 \\ 0 \\ -5/(18\sqrt{3}) \\ 1/2 \end{pmatrix}.$$

The eigenvalues of the matrix T_{-1} are defined by the relations

$$\lambda_{1,2} = (17 \pm \sqrt{433})/18, \quad \lambda_3 = 5/3, \quad \lambda_4 = 1/3, \quad \lambda_5 = 1/9. \tag{4.5}$$

We see that all the eigenvalues of Q_1 are non-integer. Hence according to Proposition 1.1 we obtain the assertion.

Proposition 4.1 *Let ρ be integer.The KZ system in the case (n=5,m=5) has no rational solutions.*

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